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Some Characterizations Of Ternary Semigroups

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Abstract-- F.M. Sison gave the idea of regular ternary semigroups. Recently, M. L. Santiago and S. Bala developed the theory of ternary semigroups. In this paper we proved some properties of ternary semigroups and mainly we proved that "A lateral zero ternary semigroup T is regular if and only if T is an idempotent ternary semigroup" and "A quasi commutative ternary semigroup T is a commutative ternary semigroup if all elements of T are idempotent".

Index Terms-- Ternary semigroup, idempotent, left(lateral and right) zero, left(lateral, right, compleatly and intra) regular, commutative, quasi commutative, pseudo commutative, normal.

1. INTRODUCTION

The theory of ternary algebraic system was introduced by Lehmer [4] in 1932, but earlier such structures were studied by Kasner who gave the idea of n-ary algebras. Ternary semigroups are universal algebras with one associative ternary operation.

In 1955, J. Los studied some properties of ternary semigroups and proved that every ternary semigroup can be embedded in a semigroup. F.M. Sison [7] introduced the notion of regular ternary semigroup. In 1983, M.L. Santiago developed the theory of ternary semigroups and semiheaps. T. K. Dutta, S. Kar and B. K. Maity [1] studied some properties of regular ternary semigroup, completely regular ternary semigroup, intra-regular ternary semigroup and characterized them by using various ideals of ternary semigroups. Many results in ordinary semigroups may be extended to n-ary semigroups for arbitrary n but the transition from n = 3 to arbitrary n entails a great degree of complexity that makes it undesirable for exposition. This section contains preliminary concepts and Basic results of ternary semigroups.

2. PRELIMINARY CONCEPTS AND RESULTS

2.1Definition: A class K with an operation between triplets of elements is called a triplex if the following postulates hold.

Postulate I. $(a.b.c)d.e = d.(a.b.c).e = d.e(a.b.c)$

$$= (a.b.d).c.e = (a.b.e).c.d = (a.c.d)b.e$$

$$= (a.c.e).b.d = (a.d.e).b.c = (b.c.d).a.e$$

$$= (b.c.e).a.d = (b.d.e).a.c = (c.d.e).a.b$$

provided a,b,c,d,e and all the expressions belongs to K.

Postulate II. If $a,b,c \in K$, then there is an element x of K such that $a.b.x = c.$

The number of elements in K is called the order of triplex and is specified when necessary by adding one of the postulates:

Postulate III_1. K contains 'n' elements.

Postulate III_2. K contains infinitely many elements.

According as III_1 or III_2 holds, the triplex is called finite or infinite.

Examples:

1. Set of all natural numbers is a triplex under usual multiplication.
2. $Z = \{\pm 1, \pm 2, \pm 3, ...\}$ is a triplex under usual multiplication.
3. Let M be the set of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ of order 2×2 matrices for a and b are natural numbers. Then M is also a triplex under usual matrix multiplication .

2.2 Definition: A non-empty set T is said to be ternary semigroup if there exists a ternary operation $\cdot : T \times T \times T \to T$ written as $(a,b,c) \to a.b.c$ satisfies the following identity $(a.b.c).d.e = a.(b.c.d).e = a.b.(c.d.e)$ for any $a,b,c,d,e \in T$.

Example 1: Let T $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Then T is a ternary semigroup under usual multiplication.

Example 2: Let $T = \{0,1,2,3,4,5\}$ and $abc = (a*b)*c$ for all $a,b,c \in T$ where '$*$' is defined in the following table

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*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	2	1	1
3	0	1	1	1	2	2
4	0	1	4	5	1	`1
5	0	1	1	1	4	5

Then $(T, *)$ is a ternary semigroup.

Example 3: Let Z^- be the set of all negative integers. Then Z^- is a ternary semigroup with the usual ternary multiplication.

Note: 1. In generally $a.b.c$ will write as abc.
2.Any semigroup can be reduced to a ternary semigroup.
3. Every ternary semigroup need not be a semigroup.

For example, let $T=\{-i,i\}$ is a ternary semigroup under the complex multiplication '.' . But it is not a semigroup by the following table

.	$-i$	i
$-i$	-1	1
i	1	-1

2.3 Definition: A ternary semigroup T is said to be commutative if $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in T$.

Example: $T=\{0,-i,i\}$ be a commutative ternary semigroup with the usual ternary multiplication.

2.4 Definition: A ternary semigroup T is said to be quasi commutative if for any $a,b,c \in T$, there exists a natural number 'n' such that $abc = b^n ac = bca = c^n ba = cab = a^n cb$.

2.5 Definition: A ternary semigroup T is said to be normal if $abT = Tab$ for all $a,b \in T$.

2.6 Definition: A ternary semigroup T is said to be left pseudo commutative if $abcde = bcade = cabde = bacde = cbade = acbde$ for all $a,b,c,d,e \in T$.

2.7 Definition: A ternary semigroup T is said to be lateral pseudo commutative if $abcde = acdbe = adbce = acbde = adcbe = abdce$ for all $a,b,c,d,e \in T$

2.8 Definition: A ternary semigroup T is said to be right pseudo commutative if $abcde = abdec = abecd = abdce = abedc = abced$ for all $a,b,c,d,e \in T$.

Example: Let $T=\{a,b,c,d,e\}$ be a set. Define a ternary operation '.' on T. where '.' is defined by the following table

.	a	b	c	d	e
a	a	a	a	a	a
b	b	a	a	a	a
c	a	a	a	a	a
d	a	a	a	a	a
e	a	b	c	d	e

Then $(T, .)$ is a left , lateral and right pseudo commutative ternary semigroup.

2.9 Definition: A ternary semigroup T is said to be pseudo commutative if T is left, lateral and right pseudo commutative ternary semigroup.

2.10 Definition: An element 'a' of a ternary semigroup T is said to be left (lateral , right) identity if $aat = t$ ($ata = t$, $taa = t$) for all $t \in T$.

2.11 Definition: An element 'a' of ternary semigroup T is said to be identity or unital if $aat = ata = taa = t$ for all $t \in T$.

Example: Let Z_0^- be the set of all non-positive integers. Then with the usual ternary operation '.', Z_0^- forms a ternary semigroup with the identity element -1.

2.1 Theorem[6]: If 'a' is left identity, b is lateral identity and c is right identity of a ternary semigroup T then $a=b=c$.

2.2 Theorem[6]: Any ternary semigroup T has atmost one identity.

Note: The identity of ternary semigroup is usually denoted by '1' (or) 'e'.

2.12 Definition: An element 'a' of a ternary semigroup T is said to be left zero (lateral zero, right zero) of T if $abc = a$ ($bac = a$, $bca = a$) for all $b,c \in T$.

A ternary semigroup T is said to be left (lateral, right) zero ternary semigrouop if every element of T is left (lateral, right) zero element.

2.13 Definition: An element 'a' of a ternary semigroup T is said to be zero of T if $abc = bac = bca = a$ for all $b,c \in T$.

A ternary semigroup T is said to be zero ternary semigrouop if every element of T is zero element.

Example: Let $0 \in T$ and $|T| > 2$. Then T with the ternary operation '.' defined by $x.y.z = \begin{cases} x & \text{if } x = y = z \\ 0, & \text{otherwise} \end{cases}$ is ternary semigroup with 0 (zero).

2.3 Result[6]: If a is left zero, b is lateral zero and c is right zero of a ternary semigroup T then $a = b = c$.

2.4 Result[6]: Any ternary semigroup T has at most one nonzero element.

2.14 Definition: An element 'a' of ternary semigroup T is said to be an idempotent if $a^3 = a$.

Note: The set of all idempotent elements in a ternary semigroup T is denoted by $E(T)$.

2.15 Definition: An element 'a' of ternary semigroup T is said to be a proper idempotent element provided 'a' is an idempotent and which is not an identity of T when identity exists .

2.16 Definition: A ternary semigroup T is said to be an idempotent ternary semigrouop or a ternary band if every element of T is an idempotent.

2.17 Definition: An element 'a' of a ternary semigroup T is said to be regular if there exists $x, y \in T$ such that $axaya = a$.

A ternary semigroup T is said to be regular ternary semigroup if every element of T is regular.

2.1 Theorem: Every idempotent element of a ternary semigroup T is regular.

Proof: Let T is a ternary semigroup and let every element of a ternary semigroup T is idempotent. Then $a^3 = a$ for all $a \in T$.

Now to prove that every idempotent element of a ternary semigroup T is regular.

Consider $a = a^3$ implies that

$a = a^3 = aaa = aa^3a = aaaaa = aa^3aa^3a$

$= aa^3a^3a^3a = aa^3aaaaa^3a$

$= aa^5aa^3a = axaya$ (put $x = a^5$ and $y = a^3$). Hence a is regular in T . Therefore every idempotent element of T is regular.

2.18 Definition: An element 'a' of a ternary semigroup T is said to be left (lateral, right) regular if there exists $x, y \in T$ such that $a = a^3xy$ $(a = xa^3y, a = xya^3)$.

2.19 Definition: An element 'a' of a ternary semigroup T is said to be intra regular if there exists $x, y \in T$ such that $a = xa^5y$.

2.20 Definition: An element 'a' of a ternary semigroup T is said to be completely regular if there exists $x, y \in T$ such that $axaya = a$ and $axa = aax = xaa = aya = aay = yaa$ $= axy = yxa = xay = xya$.

A ternary semigroup T is said to be completely regular ternary semigroup if every element of T is completely regular.

2.2 Theorem: Let T be a ternary semigroup and $a \in T$. If 'a' is a completely regular element in T then 'a' is left regular, lateral regular and right regular in T .

Proof: Let T be a ternary semigroup and $a \in T$. Let 'a' is a completely regular element in T. Then $axaya = a$ and $axa = aax = xaa = aya = aay = yaa$ $= axy = yxa = xay = xya$.

Consider $a = axaya$ implies that $a = axaya = aaxya = aaaxy = a^3xy$ implies that $a = a^3xy$.

Hence a is left regular.

Similarly we can proved a is right(lateral) regular.

2.3 Theorem: If T is a commutative ternary semigroup then T is a quasi commutative ternary semigroup.

Proof: Let T is a commutative ternary semigroup. Then $abc = bca = cab = bac = cba = acb$ for all $a, b, c \in T$

(1)

We have to prove that T is a quasi commutative ternary semigroup. i.e., For any $a, b, c \in T$, there exists a natural number 'n' such that

$abc = b^nac = bca = c^nba = cab = a^ncb$.

(2)

From (1), the (2) is exists for n=1.

Hence T is a quasi commutative ternary semigroup.

2.5 Result[6] : If T is a quasi commutative ternary semigroup then T is a normal ternary semigroup.

2.6 Result[6]: Every commutative ternary semigroup T is a normal ternary semigroup.

2.7 Result: If T is a commutative ternary semigroup then T is a pseudo commutative ternary semigroup.

2.4 Theorem: A lateral zero ternary semigroup T is regular if and only if T is an idempotent ternary semigroup.

Proof: Let T be a lateral zero ternary semigroup. Then an element $a \in T$ such that $bac = a$ for all $b, c \in T$.

Suppose T is a regular ternary semigroup. Then for any $a \in T$, there exists $x, y \in T$ such that $axaya = a$.

To prove that T is an idempotent ternary semigroup

Consider $a = axaya$

$= aaa \quad (\because xay = a \ )$

$= a^3$

Hence $a^3 = a$ for all $a \in T$.

Therefore T is an idempotent ternary semigroup.

Conversely, assume that T is an idempotent ternary semigroup. Then $a^3 = a$ for all $a \in T$.

To prove that T is a regular ternary semigroup

Consider $a = a^3$ implies that $a = a^3 = aa^2 = a^3a^2 = aaaaa = aa^3aa^3a \Rightarrow a$ is regular.

Therefore T is a regular ternary semigroup.

2.5 Theorem: A quasi commutative ternary semigroup T is a commutative ternary semigroup if all elements of T are idempotent.

Proof: Let T be a quasi commutative ternary semigroup. Then

$abc = b^nac = bca = c^nba = cab = a^ncb$ for all $a,b,c \in T$, where 'n' is a natural number (1)

Since $a \in T \Rightarrow a^3 = a$

$\Rightarrow aaa^3 = aaa \Rightarrow a^5 = a^3 = a$

$\Rightarrow a^5 = a$, $a^7 = a$,....

In generally we write this $a^{2n+1} = a$ for $n = 1,2,3,...............$

From theorem 2.1, every idempotent element of T is regular.

Here 'a' is regular then there exists $x, y \in T$ such that $a = axaya$.

To prove that T is commutative.

i.e., $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in T$. From equation (1) it is enough to prove that

$b^nac = bac$, $c^nba = cba$ and $a^ncb = acb$ for $n = 1,2,3,...............$ Consider $b^nac = bb^{n-1}ac$

$= b^{2n+1}b^{n-1}ac$ ($\because b = b^{2n+1}$)

$= bb^{3n-1}ac$,

$= bb^{3n-2}bac$

$= bb^{3n-2}b^{2n+1}ac$

$= bb^{3n-2}bb^{2n}ac$

$= bb^{3n-2}bb^{2n-1}bac$

$= bxybac$

(where $x = b^{3n-2}$ and $y = b^{2n-1}$)

$= bac$ (Since 'b' is regular)

$b^nac = bac$.

Similarly,

$c^nba = cba$ and $a^ncb = acb$.

Therefore $abc = bca = cab = bac = cba = acb$ for all $a,b,c \in T$.

Hence T is a commutative ternary semigroup.

2.6 Theorem: A regular ternary semigroup T is lateral zero ternary semigroup if and only if T is an idempotent ternary semigroup.

Proof: Let T be a regular ternary semigroup. Then for any $a \in T$, there exist $x, y \in T$ such that $axaya = a$.

Suppose T is lateral zero ternary semigroup. Then for any element $a \in T$ such that $bac = a$ for all $b, c \in T$.

To prove that T is an idempotent ternary semigroup.

Consider $a = axaya$

$= aaa$ ($\because xay = a$)

$= a^3$

$a^3 = a$ for every $a \in T$.

Hence T is an idempotent ternary semigroup.

Conversely, assume that T is an idempotent ternary semigroup. Then $a^3 = a$ for all $a \in T$.

To prove that T is lateral zero ternary semigroup

Consider $a = a^3$ implies that

$a = a^3 = aaa = aa^3a = aaaaa$

$= aaa^3aa = a^5aa^3 \Rightarrow a = a^5aa^3$ (1)

Let $b = a^5$ and $c = a^3$. Then from (1) $a = bac$ for all $b, c \in T$.

Therefore T is lateral zero ternary semigroup.

3. References:

[1]. T.K. Dutta, , S. Kar, and B.K. Maity, "On ideals in regular ternary semigroups", Discuss. Math.Gen. Algebra Appl. 28 (2008), 147-159.

[2]. J.M. Howie, "An introduction to semigroup theory", academic press, 1976.

[3]. D.H. Lehmer, "A ternary analogue of abelian groups" , Amer. J. Math. 54 (1932) 329-338.

[4]. M.L. Santiago, "Some contributions to the study of the ternary semigroups and semiheaps", (Ph.D. Thesis, 1983, University of Madras).

[5]. M.L. Santiago, and S.S. Bala, "Ternary semigroups", Semigroup Forum, 81 (2010) 380-388.

[6]. Y. Sarala, A. Anjaneyulu, and D. Madhusudhana Rao, "Ternary semigroups", International Journal of Mathematical Sciences, Technology and Humanities 76 (2013) 848-859.

[7]. F.M. Sison, "Ideal theory in ternary semigroups", Math. Japon, 10 (1965), 63-84.